Parabolic Approximation for Sound Propagation in the Atmosphere

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Propagation of sound in a stratified moving medium is discussed through an extension of the parabolic approximation to the acoustic equations of motion for short wavelengths. The parabolic approximation is related to the theory of geometric acoustics, and it is shown that it yields an improvement in accuracy over geometric theory. Also, the approximation corrects cumulative failures of geometric theory which occur when sound propagates many wavelengths from its source. The theory is illustrated by application to simple examples of quasiplane wave propagation.

I. Introduction

RECENTLY there has been an increased interest in the study of sound propagation in the atmosphere, particularly in connection with federal regulations limiting aircraft noise on landing and takeoff. Current prediction schemes associated with these regulations are based largely on geometric acoustics theory as it applies to homogeneous stationary media. At all but the very lowest frequencies, this theory is perhaps the only relatively simple analytical technique available to handle problems of sound propagation. However, it suffers from numerous deficiencies, as is commonly known. For example, in its simplest form, geometric theory does not account for diffraction effects and leads to results which are not uniformly valid descriptions of the sound field over the region in which propagation occurs. Even in the absence of diffraction, geometric acoustics is only a leading term in an asymptotic expansion of the sound field for large values of a dimensionless wavenumber, k. The first term satisfies the equations of acoustic motion only up to terms of $0(1)$ in k; a second term may improve the accuracy to $0(1/k)$, but in general the improved approximation does not remain uniformly valid for propagation through inhomogeneous and/or moving media as the sound travels through distances from its source of order k. Cumulative effects on the wave structure, which arise from variations in the ambient properties of the medium or from the source data in directions transverse to the geometric acoustics rays, are not properly accounted for by a two-term geometric acoustics approximation. The present paper is concerned with an approximate theory which improves upon geometric acoustics to yield an $0(1/k)$ approximation that remains valid for long distance propagation.

In this paper, the authors present some results of a current study of the parabolic approximation to the acoustic equations of motion and its application to sound propagation in inhomogeneous moving media. The approximation has been used extensively in the past, but only for propagation in stationary media having slowly varying inhomogeneities. Many examples exist in the Russian literature of its application to diffraction phenomena.[1,2] Also, it is currently being widely used in problems of underwater acoustics involving simple geometries and media with slowly varying ambient properties.[3,4] Until recently, the justification for the approximation appears to have been largely on intuitive grounds, and it has not been extended to problems of fully three-dimensional propagation or to propagation in moving media. However, a recent paper by Kriegsmann and Larsen[5] presents a new interpretation of the parabolic approximation, relating the technique explicitly to geometric theory and extending its validity to problems which are essentially multidimensional, and which are not restricted to slowly varying inhomogeneities. The work of Kriegsmann and Larsen applies to the reduced wave equation, and thus does not take account of the effect of motion of the medium on the propagated sound. However, the formal connection between the geometric theory and the parabolic approximation which they establish provides a key to the analogous approximation for moving media. It is the purpose of this paper to develop the parabolic approximation for horizontally stratified media moving unidirectionally in horizontal direction according to a specified vertically sheared flow.

In order to review briefly the concepts of geometric theory and the work of Ref. 5, consider a medium in which the acoustic pressure p can be taken to satisfy a wave equation with sound speed c dependent upon the spatial coordinates. If solutions are assumed of the form $p=P\exp(ik\theta-i\omega t)$, then according to geometric theory, for sufficiently large k, θ satisfies the eikonal equation

$$(\nabla\theta)^2=[c_0/c(r)]^2 \tag{1}$$

where $c_0=\omega/k$ is a representative sound speed, and $P(r;k)$ satisfies the equation

$$ik(2\nabla P\cdot\nabla\theta+P\nabla^2\theta)+\nabla^2P=0 \tag{2}$$

If $P(r;k)$ is assumed to be expressible in a series of inverse powers of k, then to leading order $P=\psi(r)$ where $\psi(r)$ satisfies

$$2\nabla\psi\cdot\nabla\theta+\psi\nabla^2\theta=0 \tag{3}$$

which is the transport equation of geometric acoustics. It is equivalent to a first-order ordinary differential equation along the rays of geometric acoustics which, for the stationary medium, are the curves orthogonal to the surfaces $\theta=$const. Alternatively, if it is noted that Eq. (3) is equivalent to $\nabla\cdot(\psi^2\nabla\theta)=0$, an application of the divergence theorem to a tube of rays yields the familiar result that the energy flow remains constant along a ray tube. The lowest order geometric approximation $p\simeq\psi\exp(ik\theta-i\omega t)$ is generally not uniformly valid because of its failure to account for the ∇^2P in Eq. (2), the terms of which are generally not everywhere negligible compared to the left member even for large values of k.

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The theory of Ref. 5 proceeds by writing $p(r;k) = \psi(r)A(r;k)$, where $\psi(r)$ is taken to satisfy the transport equation, Eq. (3). Then $A(r;k)$ is expressed in terms of new independent variables (θ,ξ,η), where $\theta=$const. satisfies Eq. (1) and thus is a coordinate along the geometric ray direction, while ξ and η are transverse orthogonal coordinates normal to the rays. Substitution of the form $\psi(r)A(r;k)$ into Eq. (2) and transformation of the independent variables to (θ,ξ,η) then yields the equation[5]

$$2ik\frac{\partial A}{\partial\theta}+\nabla_T^2A+\frac{2}{\psi}\nabla_T\psi\cdot\nabla_TA+\frac{1}{\psi}\nabla^2\psi+0\left(\frac{1}{k}\right)=0 \quad (4)$$

where ∇_T is the transverse gradient operator in the ξ, η coordinates. Neglecting the $0(1/k)$ terms in Eq. (4) yields the parabolic approximation

$$2ik\frac{\partial A}{\partial\theta}+\nabla_T^2A+\frac{2}{\psi}\nabla_T\psi\cdot\nabla_TA+\frac{1}{\psi}\nabla^2\psi=0 \quad (5)$$

Equation (5) is Kriegsmann and Larsen's parabolic equation for A which, assuming ψ is sufficiently smooth, yields an approximate solution to the wave equation in which only terms of $0(1/k)$ have been neglected. The advantage in using Eq. (5) lies in its parabolic nature—it can be solved numerically by marching in the θ direction in an extremely simple and efficient manner. The function A describes $0(1/k)$ changes in p along a geometric ray, and thus the approximation $p\simeq\psi A\,\exp(ik\theta-i\omega t)$ is more accurate than the geometric theory. In addition, the improved approximation accounts both for diffraction effects and for cumulative breakdowns of the geometric theory at large distances from the sound source. At a caustic, of course, the function ψ is singular.[6] Thus the parabolic approximation ψA cannot be expected to remain uniformly valid in the vicinity of caustics.

In the following sections, a corresponding approximation is carried out for moving stratified media, and some examples of its application are discussed.

II. Formulation

Consider a z-stratified inviscid nonconducting ideal gas in steady horizontal motion along the x direction at velocity $U(z)$. The total fluid pressure, density, and velocity in the medium are written as $p_0(z)+p$, $\rho_0(z)+\rho$, and $(U+u, v, w)$, respectively, where p, ρ, and $(u,v,w)=\boldsymbol{u}$ are small perturbations on the basic steady flow. The linearized acoustic equations of motion are easily derived from the principles of conservation of mass, linear momentum, and energy in the form

$$D_0\rho/Dt+\rho_0\nabla\cdot\boldsymbol{u}+\rho_0'w=0 \quad (6a)$$

$$\rho_0D_0u/Dt+\rho_0U'w+p_x=0 \quad (6b)$$

$$\rho_0D_0v/Dt+p_y=0 \quad (6c)$$

$$\rho_0D_0w/Dt+p_z+g\rho=0 \quad (6d)$$

$$D_0p/Dt+p_0'w-c^2(D_0\rho/Dt+\rho_0'w)=0 \quad (6e)$$

In Eqs. (6) D_0/Dt is the linearized material derivative operator $\partial/\partial t+U(z)\partial/\partial x$, $c^2(z)=\gamma p_0/\rho_0$ is the speed of sound in the stratified medium, and g is the acceleration of gravity. The primes denote differentiation of the steady flow quantities with respect to z, and the subscript notation is used for partial derivatives of the acoustic quantities with respect to the space variables. The ambient pressure gradient p_0' is equal to $-\rho_0g$.

Substitution of Eq. (6e) into Eq. (6a) and operation with D_0/Dt yields

$$D_0^2p/Dt^2+p_0'D_0w/Dt+\rho c_0^2D_0(\nabla\cdot\boldsymbol{u})/Dt=0 \quad (7)$$

while differentiation of Eqs. (6b-6d) with respect to x,y, and z, respectively, gives

$$\rho_0D_0(\nabla\cdot\boldsymbol{u})/Dt=-\nabla^2p-2\rho_0U'w_x-g\rho_z-\rho_0'D_0w/Dt$$

If this relation is substituted into Eq. (7) it becomes

$$D_0^2p/Dt^2-c^2\nabla^2p-2\rho_0c^2U'w_x-c^2g\rho_z$$
$$-(\rho_0g+c^2\rho_0')D_0w/Dt=0 \quad (8)$$

Equation (8) in conjunction with Eqs. (6d) and (6e), constitutes the system of three equations for $p(r,t)$, $\rho(r,t)$, and $w(r,t)$ which govern the propagation of sound in a stratified atmosphere in unidirectional, vertically sheared horizontal flow.

It is commonly assumed that the predominant effect of gravity on the acoustic field is to maintain the stratified base state $p_0(z)$, and that the perturbation of the gravity force per unit mass resulting from density changes associated with the acoustic field is of negligible effect on the sound.[6] This is a valid supposition for ordinary acoustic wavelengths, provided that the propagation distance is sufficiently small compared to the scale length over which the ambient pressure varies. The effect of the assumption is to neglect the terms $\rho_0'w$, $p_0'w$, and $g\rho$ in the system, Eqs. (6). This approximation will be adopted in the present work in order to simplify the algebraic details. It should be emphasized, however, that it is not necessary to do so; all of the theory to be derived can be carried through for the complete set of Eqs. (6d, 6e) and (8). Neglecting the terms just indicated, however, removes the density perturbation ρ from Eqs. (8) and (6d), which become

$$D_0^2p/Dt^2-c^2\nabla^2p-2\rho_0c^2U'w_x+c^2(\rho_0'/\rho_0)p_z=0 \quad (9)$$

and

$$\rho_0D_0w/Dt+p_z=0 \quad (10)$$

The approximate equations (9) and (10) will be assumed here to represent with sufficient accuracy the governing equations for sound propagation in a stratified moving medium under the action of gravity.

Solutions to the system (9) and (10), are sought in the form

$$p=P(r;k)\exp[ik\theta(r)-i\omega t]$$
$$w=W(r;k)\exp[ik\theta(r)-i\omega t]$$

where $k=\omega/c_0$, and c_0 is a representative value of the sound speed. Substitution of these expressions into Eq. (9) yields

$$(ik)^2\left[(\nabla\theta)^2-\frac{q^2}{c^2}\right]P+ik\left[2\nabla P\cdot\nabla\theta+2\frac{Mq}{c}P_x\right.$$
$$\left.+\left(\nabla^2\theta-M^2\theta_{xx}-\frac{\rho_0'\theta_z}{\rho_0}\right)P+2\rho_0U'\theta_xW\right]+\nabla^2P-M^2P_{xx}$$
$$-\frac{\rho_0'}{\rho_0}P_z+2\rho_0U'W_x=0 \quad (11)$$

and Eq. (10) becomes

$$ik[\theta_zP-\rho_0qW]+P_z+\rho_0UW_x=0 \quad (12)$$

In Eqs. (11) and (12), the expression q has been introduced by defining

$$q=c_0-U\theta_x=c(z)[N-M\theta_x] \quad (13)$$

where $N(z)=c_0/c(z)$ is the index of refraction of the medium and $M(z)=U(z)/c(z)$ is the Mach number of the steady flow. Now it is convenient to replace the term involving

ikW in Eq. (11) by its equivalent from Eq. (12). Then Eq. (11) becomes

$$(ik)^2\left[(\nabla\theta)^2-\frac{q^2}{c^2}\right]P+ik\left[2\nabla P\cdot\nabla\theta+2\frac{Mq}{c}P_x\right.$$
$$\left.+\left(\nabla^2\theta-M^2\theta_{xx}-\frac{\rho_0'}{\rho_0}\theta_z+2\frac{U'}{q}\theta_x\theta_z\right)P\right]$$
$$+\nabla^2P-M^2P_{xx}+\left(\frac{2U'\theta_x}{q}-\frac{\rho_0'}{\rho_0}\right)P_z+2\frac{\rho_0c_0}{q}U'W_x=0 \quad (14)$$

Using Eq. (13) and the relation $2c'/c=p_0'/p_0-\rho_0'/\rho_0$, the quantity $2U'\theta_x/q-\rho_0'/\rho_0$ can be written as

$$2U'\frac{\theta_x}{q}+2\frac{c'}{c}-\frac{p_0'}{p_0}=2\frac{c}{q}(M'\theta_x-N')-\Pi'$$

where $\Pi=\ln p_0$. Hence, the system of equations for P and W can be written finally as

$$(ik)^2\left[(\nabla\theta)^2-\frac{q^2}{c^2}\right]P+ik\left\{2\nabla P\cdot\nabla\theta+2\frac{Mq}{c}P_x\right.$$
$$\left.+\left[\nabla^2\theta-M^2\theta_{xx}+2\frac{c}{q}\left(M'\theta_x-N'\right)\theta_z-\Pi'\theta_z\right]P\right\}$$
$$+\nabla^2P-M^2\theta_{xx}+\left[2\frac{c}{q}\left(M'\theta_x-N'\right)-\Pi'\right]P_z$$
$$+2\frac{\rho_0c_0}{q}U'W_x=0 \quad (15a)$$

and

$$ik[\rho_0qW-\theta_zP]-P_z-\rho_0UW_x=0 \quad (15b)$$

In the following sections of this paper, asymptotic solutions to the system Eqs. (15) will be considered for large values of the parameter k. For this purpose, it is convenient to introduce a characteristic length L which can be taken to be the smaller of the scale lengths associated with the variations in $M(z)$, $N(z)$, and $\Pi(z)$. Then if the independent variables (x,y,z) are scaled with respect to L, Eqs. (15) remain identical in form, except that k becomes the dimensionless kL. Without introducing new notation, it is assumed that this scaling has been carried out and that k,x,y,z in Eqs. (15) are in dimensionless form.

Geometric acoustic theory consists of the assumptions that, for large values of k, Eqs. (15) imply that

$$(\nabla\theta)^2-q^2/c^2=0 \quad (16)$$

and that $P(r;k)$ and $W(r;k)$ can be expanded in series of inverse powers of k. Then, to leading order, $p\simeq\psi(r)$, $W\simeq(\theta_z/\rho_0q)\psi$, where ψ satisfies

$$2\nabla\psi\cdot\nabla\theta+2\frac{Mq}{c}\psi_x+\left[\nabla^2\theta-M^2\theta_{xx}\right.$$
$$\left.+\frac{2c}{q}\left(M'\theta_x-N'\right)\theta_z-\Pi'\theta_z\right]\psi=0 \quad (17)$$

Equation (16) is the eiconal equation and Eq. (17) is the first-order transport equation of geometric acoustic theory. In the following it will be assumed that the phase θ and amplitude ψ are known, and the reader is referred to detailed discussions of the geometric theory in the literature.[6-8] Lighthill,[8] in particular, has given a valuable review of geometric theory as applied specifically to sound propagation in moving media. It is a matter of straightforward algebra to show that if Eq. (17) is multiplied by $c_0c^2\psi/\gamma p_0q^2$, it can be written as

$$\nabla\cdot\left[\frac{c_0\psi^2(\nabla\theta+(Mq/c)\boldsymbol{e}_x)}{\rho_0q^2}\right]=0$$

where $\boldsymbol{e}_x$ is a unit vector along x. If $\boldsymbol{n}$ is defined as a unit vector normal to the surfaces of constant phase, $\boldsymbol{n}=\nabla\theta/|\nabla\theta|$, then the expression inside the brackets is equivalent to

$$E=(c_0\psi^2/\rho_0q^2)\,|\nabla\theta|\,(\boldsymbol{n}+M\boldsymbol{e}_x)=(c_0\psi^2/\rho_0qc^2)\,(c\boldsymbol{n}+U\boldsymbol{e}_x)$$

This, of course, is the geometric acoustics energy flux vector,[7,8] and thus Eq. (17) implies that $|E|A_r$ is constant along ray tubes where A_r is the local cross-sectional area of the tube.

The lowest order geometric theory as embodied in Eqs. (16) and (17) accounts for $0(k)$ and $0(1)$ variations in p and w along the rays. A second term in the series for P will describe $0(1/k)$ changes in p and w along the rays, but it is well known that such a two-term geometric expansion fails in several important respects. For example, it fails to account for diffraction effects when they exist, and also it is generally not uniformly valid for propagation over distances of the order of k from the sound source. In the following section an improved approximation is discussed which includes $0(1/k)$ variations of p and w and also accounts for diffraction and for the cumulative nonuniformity present in the two-term geometric acoustics approximation.

III. The Parabolic Approximation

In the system (15), let θ satisfy the eiconal equation (16) and set

$$P(r;k)=\psi(r)A(r;k) \quad (18)$$

Substitution into Eqs. (15) yields

$$2ik\left[\left(\nabla\psi\cdot\nabla\theta+\frac{Mq}{c}\psi_x\right)A+\left(\nabla A\cdot\nabla\theta+\frac{Mq}{c}A_x\right)\psi\right]$$
$$+ik\left[\nabla^2\theta-M^2\theta_{xx}+\frac{2c}{q}\left(M'\theta_x-N'\right)\theta_z-\Pi'\theta_z\right]\psi A$$
$$+A\nabla^2\psi+2\nabla A\cdot\nabla\psi+\psi\nabla^2A-M^2(A\psi_{xx}+2A_x\psi_x+\psi A_{xx})$$
$$+\left[\frac{2c}{q}\left(M'\theta_x-N'\right)-\Pi'\right]\left(A\psi_z+\psi A_z\right)+2\frac{\rho_0c}{q}U'W_x=0 \quad (19)$$

and

$$ik(\rho_0qW-\theta_z\psi A)-\psi_zA+\psi A_z-\rho_0UW_x=0 \quad (20)$$

Now, if ψ is required to be a solution of the first-order transport equation (17), then the terms multiplying A in the first two square brackets in Eq. (19) are eliminated. This yields the equation on $A(r;k)$ in the form

$$2ik\left[\nabla A\cdot\nabla\theta+\frac{Mq}{c}A_x\right]+\nabla^2A+\frac{2}{\psi}\nabla A\cdot\nabla\psi+\frac{\nabla^2\psi}{\psi}A$$
$$-M^2\left[A_{xx}\frac{2}{\psi}\psi_xA_x+\frac{\psi_{xx}}{\psi}A\right]+\left[\frac{2c}{q}\left(M'\theta_x-N'\right)-\Pi'\right]$$
$$\times\left(A_z+\frac{\psi_z}{\psi}A\right)+\frac{2\rho_0cU'}{q\psi}W_x=0 \quad (21)$$

The first term in Eq. (21) is $\nabla A\cdot\boldsymbol{m}$, where $\boldsymbol{m}$ is the vector $\nabla\theta+M(q/c)\boldsymbol{e}_x$. But this vector is in the geometric acoustics ray direction.[7,8] Thus, the term $\nabla A \cdot m$ is proportional to the directional derivative of A along the geometric rays. This term is multiplied by k, which indicates that as k becomes large, the derivatives of A along the ray should be small compared to its derivatives in directions transverse to the ray. Otherwise, the first term in Eq. (21) would dominate and yield $A=$const. on the ray.

These considerations suggest that the independent variables in Eq. (21) should be transformed to a new set of variables along and transverse to the geometric rays in order to identify those terms which are negligible at large k. This process involves lengthy algebraic manipulation, and for the purpose of this paper, it will be carried out in detail only for the case of quasiplane wave propagation in the z direction. The transformation does not have to be carried out in Eq. (20), however. All terms in the bracket in Eq. (20) are 0(1), and thus for large k it must be true that

$$W=(\theta_z/\rho_0 q)\psi A \tag{22}$$

to leading order. This relation can be used to eliminate the term containing W_x in Eq. (21).

Quasiplane Waves

A special class of harmonic solutions to Eqs. (9) and (10) exists for which the wavefronts are planes normal to the z axis, and the amplitudes vary with x and y as well as with z. For such waves, the eiconal θ is a function of z alone, a fact which affords considerable simplification of the algebra involved in transforming Eq. (21). A quasiplane wave is generated, for example, by harmonic vibration of the plane $z=0$ with amplitude varying with x and y. In the quasiplane case, the eiconal equation (16) becomes

$$\mathrm{d}\theta/\mathrm{d}z=c_0/c(z)=N(z) \tag{23}$$

Before proceeding further with this case, it is necessary to discuss the boundary conditions associated with Eq. (21) for quasiplane waves. Commonly, it would be assumed that $W\exp(ik\theta)$ is given on $z=0$ [see Eq. (10)] and is independent of k. This immediately requires $\theta(0)=0$, and Eq. (22) becomes $\theta_z\psi A=\rho_0 q W$ on $z=0$, or

$$\psi A|_{z=0}=\rho_0(0)c(0)W(x,y,0) \tag{24}$$

Thus, there is an arbitrariness in the required boundary conditions on ψ and A. In what follows, ψ is chosen, without loss of generality, as a solution to Eq. (17) which assumes a constant value on the boundary $z=0$.

The transport equation, Eq. (17), for ψ becomes in the present case

$$2N\psi_z+2MN\psi_x-(\Pi'N+N')\psi=0$$

and a solution to this equation, which assumes a constant value on $z=0$, is

$$\psi(z)=[N(z)\rho_0(z)]^{1/2} \tag{25}$$

Then, substiution of Eqs. (25), (23), and (22) into Eq. (21), and making use of the relation $U'=(cM)'=c'M+cM'=c(M'-MN'/N)$ yields

$$2ikN\left[A_z+MA_x\right]+(1-M^2)A_{xx}+A_{yy}+A_{zz}+2\left(M'-\frac{MN'}{N}\right)A_x-\frac{N'}{N}A_z+\left[\frac{N''}{2N}-\frac{5(N')^2}{4N^2}-\frac{(\Pi')^2}{4}\right]A=0 \tag{26}$$

For the case under consideration, the geometric acoustics rays all lie in the planes normal to the y axis. The ray direction is given by the vector Me_x+e_z. A parameter $\xi(x,z)$ is introduced so that ξ is constant along a ray; that is, $\xi=$const on the curves $\mathrm{d}x/\mathrm{d}z=M(z)$, or

$$\xi=x-\int_0^z M(\zeta)\mathrm{d}\zeta \tag{27}$$

In addition,

$$\theta=\int_0^z N(\zeta)\mathrm{d}\zeta \tag{28}$$

from Eq. (23), and θ can be considered as a parameter measuring position along a ray. Let A in Eq. (26) be considered as a function of (θ,ξ,η), where $\eta=y$ and ξ and θ are as defined in Eqs. (27) and (28). Then

$$A_z=NA_\theta-MA_\xi,\quad A_x=A_\xi,\quad A_y=A_\eta$$

$$A_{zz}=N^2A_{\theta\theta}-2MNA_{\xi\theta}+M^2A_{\xi\xi}+N'A_\theta-M'A_\xi$$

and so on. Substituting these relations into Eq. (26), it becomes

$$2ikN^2A_\theta+A_{\xi\xi}+A_{\eta\eta}+N^2A_{\theta\theta}-2MNA_{\xi\theta}+\left(M'-\frac{MN'}{N}\right)A_\xi+\left[\frac{N''}{2N}-\frac{5(N')^2}{4N^2}-\frac{(\Pi')^2}{4}\right]A=0 \tag{29}$$

The final step in the analysis consists in recognizing that for large k Eq. (29) implies that A_θ is $0(1/k)$. Thus, $A_{\theta\theta}$ and $A_{\xi\theta}$ are $0(1/k)$ and can be neglected for large k giving

$$2ikN^2A_\theta+A_{\xi\xi}+A_{\eta\eta}+\left(M'-\frac{MN'}{N}\right)A_\xi+\left[\frac{N''}{2N}-\frac{5(N')^2}{4N^2}-\frac{(\Pi')^2}{4}\right]A=0 \tag{30}$$

Equation (30) is the parabolic approximation for the quasiplane wave case considered here. It differs from Eq. (21) only in that terms of $0(1/k)$ have been neglected. The solution for A using Eq. (30) describes lower order variations in the acoustic quantities which are not included in the simpler geometric theory of Eqs. (16) and (17). For $M(z)=0$, Eq. (20) reduces to a form equivalent to the results in Ref. 5, except that in that work the term containing ρ_0' in Eq. (9) is omitted from the outset. Generally, Eq. (30), subject to the boundary condition (24), must be solved numerically to determine A. However, it is a parabolic equation, and the numerical solution can be effected using a marching technique in the variable θ. In the next section of this paper a few simple examples of such numerical solutions are presented to illustrate the theory.

The General Case

The procedure for deriving the parabolic approximation in the case $\theta=\theta(x,y,z)$ is identical to that described for quasiplane waves. However, the details of this derivation are considerably more complicated than for the special case. In the present paper, the resulting parabolic equation will be given for completeness, but none of the details of its derivation are included.

In general, neither the eiconal θ nor the amplitude ψ can be written explicitly as a function of the coordinates (x,y,z). These must be determined from geometric theory by solving the ray equations corresponding to Eq. (16). Then ψ can be determined in parametric form along the rays from Eq. (17). If it is assumed that this has been carried out, then the independent variables in Eq. (21) are transformed to a set ξ and η, which are constant along a geometric ray, and another variable measuring position along a ray, which in the present work is taken to be θ itself. Once the full equation (21) is written in terms of ξ,η,θ, it is again possible to identify negligible terms for large k. The final parabolic equation which results is:

$$2ikN(N-M\theta_x)A_\theta + C_1(\xi)A_{\xi\xi} + C_1(\eta)A_{\eta\eta} + C_2(\xi)A_\xi + C_2(\eta)A_\eta + C_3 A = 0 \tag{31}$$

where

$$C_1(\xi) = (\nabla\xi)^2 - M^2\xi_x^2$$

$$C_2(\xi) = \nabla^2\xi - M^2\xi_{xx} + \frac{2}{\psi}\left[C_1(\xi)\psi_\xi - MN\xi_x\psi_\theta\right] + \left[\frac{2(M'\theta_x - N')}{n-M\theta_x} - \Pi'\right]\xi_z + \frac{2(NM'-MN')}{(N-M\theta_x)^2}\theta_z\xi_x$$

and

$$C_3 = \frac{1}{\psi}\left\{\nabla^2\psi - M^2\psi_{xx} + \left[\frac{2(M'\theta_x-N')}{N-M\theta_x} - \Pi'\right]\psi_z + \frac{2(NM'-MN')}{(N-M\theta_x)^2}\left[\theta_z\psi_x + \theta_{xz}\psi + \frac{M\theta_z\theta_{xx}}{N-M\theta_x}\psi\right]\right\}$$

The preceding coefficients have been written in what is apparently the most compact form possible; the geometric acoustics amplitude has been denoted as ψ, both when it is considered as a function of (x,y,z) and as a function of (ξ,η,θ). For example, ψ_ξ in Eq. (31) means $\partial\psi(\xi,\eta,\theta)/\partial\xi$, while ψ_x stands for $\partial\psi(x,y,z)/\partial x$. Obviously, even though Eq. (31) is parabolic and thus simple, in principle, to solve numerically, it is considerably more difficult to do so than in the previous special case. Solutions and solution techniques for Eq. (31) are the subject of current investigation by the authors.

IV. Results and Discussion

Two-Dimensional Radiation from a Plane Boundary

As an extremely simple example, which can be used as a check on numerical results, one can consider the problem of two-dimensional radiation from a vibrating plane located along $z=0$ into a stationary homogeneous medium ($M=0$, $N=1$, $\Pi'=0$). Then the acoustic quantities are functions of x and z only, and $w(x,0)=W_0(x)$. For this case, Eqs. (30) and (24) reduce to

$$2ikA_z + A_{xx} = 0 \quad -\infty<x<\infty, \quad 0<z$$

$$A(x,0) = \rho_0 c W_0(x) \quad -\infty<x<\infty \tag{32}$$

This boundary value problem can be solved exactly using a Fourier transform in x to yield for $p(x,z,t)$:

$$p(x,z,t) \simeq \rho_0 c\exp(ikz - i\omega t)(k/2\pi iz)^{1/2}\int_{-\infty}^{\infty} W_0(\zeta) \times \exp\frac{ik(x-\zeta)^2}{2z}\,d\zeta$$

A particularly instructive example of this solution is that for which $W_0(x)=H(-x)$, where H is the unit step function which corresponds to a semi-infinite piston located along $-\infty<x<0$ with the positive half of the plane $z=0$ a rigid baffle. In this case, there is a diffraction effect arising from the discontinuity in normal velocity at $x=0$. This is included in the parabolic approximation, as can be seen from the preceding solution, which becomes

$$p \simeq \rho_0 c\exp(ikz - i\omega t)\int_\mu^\infty \exp(i\zeta^2)\,d\zeta \tag{33}$$

where $\mu^2 = kx^2/2z$. Equation (33) can be expressed in terms of Fresnel functions, and it exhibits the familiar pattern of amplitude and phase oscillations in the vicinity of the geometric shadow boundary along the z axis.

Numerical Results for Quasiplane Waves

For the purpose of this paper, the authors have not attempted a sophisticated numerical procedure for solving Eqs. (30) and (24). Rather, a very simple explicit scheme was employed for the purpose of illustrating the theory presented in the preceding section. Equation (30) is split into its real and imaginary parts by writing $A=\alpha+i\beta$. Since the coefficients of equation (30) are functions of z, and θ depends only on z in this special case, it is convenient to use z rather than θ as the independent variable along the rays. Thus, Eq. (30) gives

$$\alpha_z + (1/2k)[f(z)\beta_{\xi\xi} + g(z)\beta_\xi - h(z)\beta] = 0$$

$$\beta_z - (1/2k)[f(z)\alpha_{\xi\xi} + g(z)\alpha_\xi - h(z)\alpha] = 0 \tag{34}$$

where the coefficients in Eq. (30) have been denoted f,g, and h. Equations (34) are approximated using forward differences for α_z, backward differences for β_z, and central differences for the transverse derivatives. This yields an explicit scheme of equations as follows (subscripts denote evaluation at the grid points indicated in Fig. 1):

$$\alpha_5 = \alpha_2 - \frac{af_2}{2kb^2}(\beta_3 - 2\beta_2 + \beta_1) - \frac{ag_2}{4kb}(\beta_3-\beta_1) + \frac{ah_2}{2k}\beta_1$$

$$\beta_5 = \beta_2 + \frac{af_5}{2kb^2}(\alpha_6 - 2\alpha_5 + \alpha_4) + \frac{ag_5}{4kb}(\alpha_6-\alpha_4) - \frac{ah_5}{2k}\alpha_5 \tag{35}$$

Starting with the given values of α and β on the boundary $z=0$, the first of Eqs. (35) is used to compute α on the second

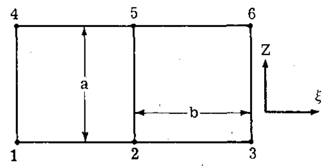

Fig. 1 Grid illustrating explicit numerical scheme of Eqs. (35).

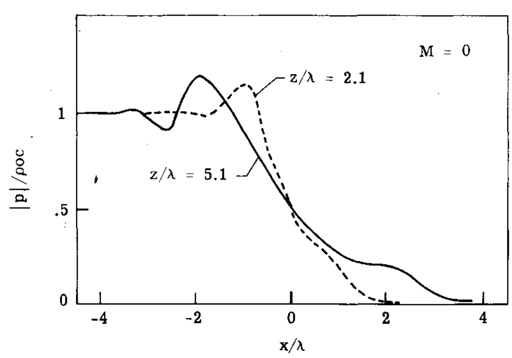

Fig. 2 Pressure magnitude from semi-infinite piston according to parabolic approximation—homogeneous, stationary medium.

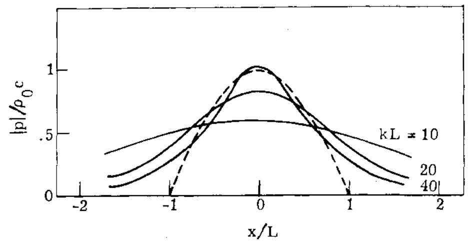

Fig. 3 Pressure magnitude according to parabolic approximation at $z/L=9.9$—homogeneous, stationary medium.

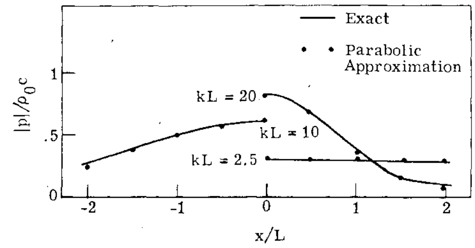

Fig. 6 Parabolic approximation vs exact solution at $z/L=9.9$—homogeneous, stationary medium.

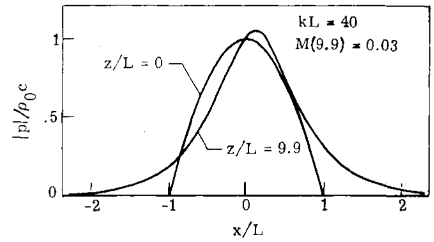

Fig. 4 Pressure magnitude according to parabolic approximation—homogeneous medium, linear flow profile.

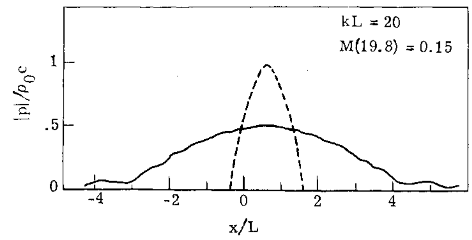

Fig. 7 Pressure magnitude according to parabolic approximation at $z/L=19.8$—homogeneous medium, linear flow profile.

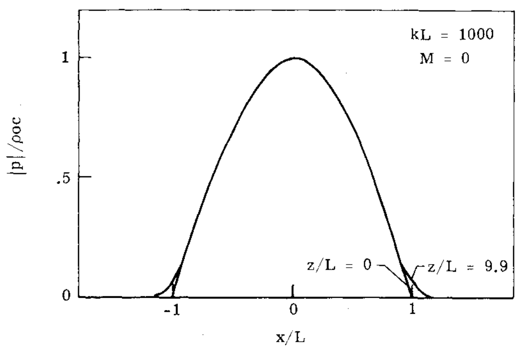

Fig. 5 Pressure magnitude according to parabolic approximation—homogeneous, stationary medium.

row. Then the second equation gives the values of β on the second row, and so on. For the computations to be discussed below the step sizes a and b in the z and ξ directions were both taken as 0.1 unit (recall that the independent variables as well as k have been assumed to be dimensionless). The scheme (35) serves quite well to illustrate the behavior of the theoretical predictions of Eq. (30).

Figure 2 shows the numerically computed pressure magnitude, using Eqs. (35), for the problem stated at the beginning of this section with $W_0=H(-x)$. Since there is no characteristic length in this problem, the result is shown as a function of x/λ, where λ is the sound wavelength. As the plot indicates, at increasing levels z the initially discontinuous solution on $z=0$ is smoothed out, with resulting diffraction bands appearing in the "illuminated" region $x<0$. The solution to the parabolic problem is given by Eq. (33), which serves as a check on scheme (35). For the cases indicated on Fig. 2, the agreement between the numerical result and the values calculated from Eq. (33) was within a few percent, except at the larger values of $|x|$ where the deviations of p from $\rho_0 c$ and zero are extremely small.

The remaining figures all describe a case in which $W_0(x)$ was taken to be $\cos(\pi x/2)$ for $-1<x<1$ and zero otherwise (x on the figures is the dimensional x and the scale length is taken as the half-width of the source region on the boundary). No temperature or gravity effects were included; that is, $N=1$, $\Pi'=0$ in all cases.

Figure 3 illustrates the numerical solution at $z/L=9.9$ for pressure magnitude at three frequencies. The initial cosine profile at $z=0$ is shown as a broken curve. This curve would also be the first-order geometric acoustics prediction at $z/L=9.9$. The transverse diffusion of the pressure signal described by the parabolic approximation is evident.

Figure 4 shows a corresponding result for $kL=40$ in the presence of a linear flow profile, $M(z)$ varying from zero at $z=0$ to 0.3 at $z=100L$. Again the transverse diffusion is evident, as is convection of the field by the basic steady flow. The shape of the curve for $z=9.9L$ is not identical to that of Fig. 3 ($kL=40$), but significant differences in shape caused by the flow gradient have not appeared at this height. It is important to note that geometric theory for this case would predict that the initial cosine profile is simply shifted along the rays given by Eq. (27).

Figure 5 is included to show the prediction of the parabolic approximation at a very high frequency, $kL=1000$. The plot shows the no-flow result in which the sound field has spread only very slightly away from the region between $x=\pm 1$. Of course, this is to be expected; for sufficiently large k, the parabolic approximation yields $A=$const. along the rays and thus reproduces geometric theory.

For the no-flow cases discussed here, the exact solution to the full Helmholtz equation governing the sound field can be written using the two-dimensional Green's function. Figure 6 gives a comparison of the exact pressure magnitude, shown in solid lines, with the parabolic approximtion at $z/L=9.9$ for three frequencies, again for the cosine boundary condition on $z=0$. The agreement between the exact and approximate results is remarkable even for k as low as 2.5. In this simple problem, the parabolic approximation differs from the exact solution by no more than 3-4% at any of the points shown.

Finally, Fig. 7 shows, for $kL=20$, the parabolic prediction for pressure magnitude at a distance $z/L=19.8$. In this case a linear flow profile with a gradient 2.5 times that used in Fig. 4 was assumed so that $M=0.15$ at $z/L=19.8$. Shown in the broken line is the initial cosine profile convected along the geometric rays, which would be the prediction of geometric theory. The large flow gradient in this case has begun to affect the shape of the parabolic prediction, which can be seen to be slightly asymmetric at this height.

The preceding results, although intended only to illustrate the nature of the parabolic approximation, suggest that the technique could prove to be a useful tool for studies of sound propagation in a moving inhomogeneous atmosphere. At frequencies of practical interest, the approximation provides a description of the sound field to a higher degree of accuracy than does geometric theory. Current studies are concerned with application of Eq. (31) to fully three-dimensional propagation and with application of accurate and efficient numerical methods (see Ref. 3) for its solution in cases of practical interest.

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